Example: Calculate $\int_{-1}^{2} (3x^2 - 1)\,dx$, using the definition.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

$$\int_{-1}^{2} (3x^2-1)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[3\left(-1+\frac{3k}{n}\right)^2 - 1\right]\frac{3}{n}$$

$\left(\Delta x = \frac{2-(-1)}{n} = \frac{3}{n},\ x_k = -1 + \frac{3k}{n}.\right)$ $f(x) = 3x^2 - 1$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[3\left(1 + -\frac{6k}{n} + \frac{9k^2}{n^2}\right) - 1\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[2 - \frac{18k}{n} + \frac{27k^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\sum_{k=1}^{n} 2 - \sum_{k=1}^{n} \frac{18k}{n} + \sum_{k=1}^{n} \frac{27k^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{6}{n}\left(\sum_{k=1}^{n} 1\right) - \frac{54}{n^2}\sum_{k=1}^{n} k + \frac{81}{n^3}\sum_{k=1}^{n} k^2\right)$$

Formulas: $\sum_{k=1}^{n} 1 = n$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \lim_{n\to\infty}\left(\frac{6}{n}\cdot n - \frac{54}{n^2}\underbrace{\left(\frac{n(n+1)}{2}\right)}_{\frac{n^2+n}{2}} + \frac{81}{n^{\cancel{3}\,2}}\underbrace{\frac{(\cancel{n})(n+1)(2n+1)}{6}}_{\frac{2n^2+3n+1}{6}}\right)$$

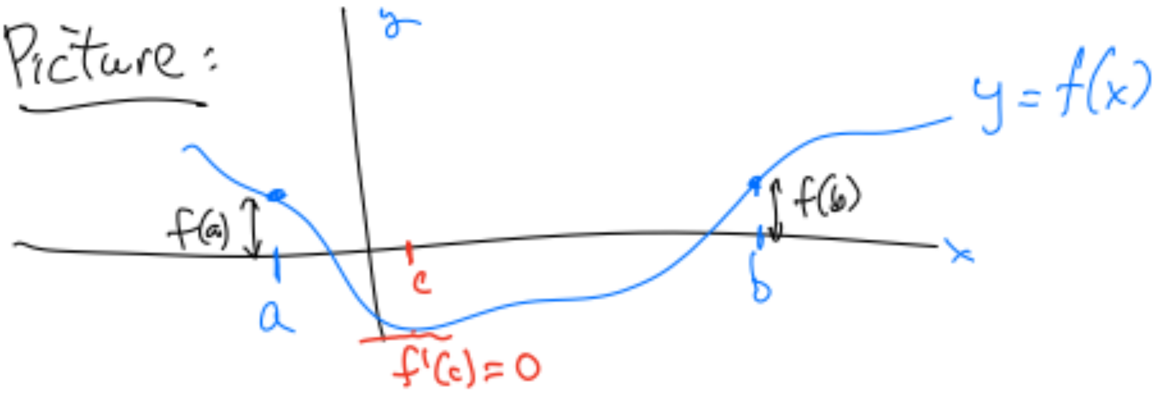
$$= \lim_{n\to\infty}\left(6 - \frac{\overset{27}{\cancel{54}}}{\cancel{2}}\cdot\underbrace{\frac{n^2+n}{n^2}}_{\underbrace{1+\frac{1}{n}}_{\to 1}} + \frac{\overset{27}{\cancel{81}}}{2\cancel{6}}\cdot\underbrace{\left(\frac{2n^2+3n+1}{n^2}\right)}_{\underbrace{2+\frac{3}{n}+\frac{1}{n^2}}_{\to 2}}\right)$$

$$= 6 - 27 + 27 = \boxed{6}.$$

Rolle's Theorem:

If $f:[a,b]\to\mathbb{R}$ is continuous and differentiable on (a,b), and if $f(a)=f(b)$, then there is a point $c\in(a,b)$ such that $f'(c)=0$.

Picture:



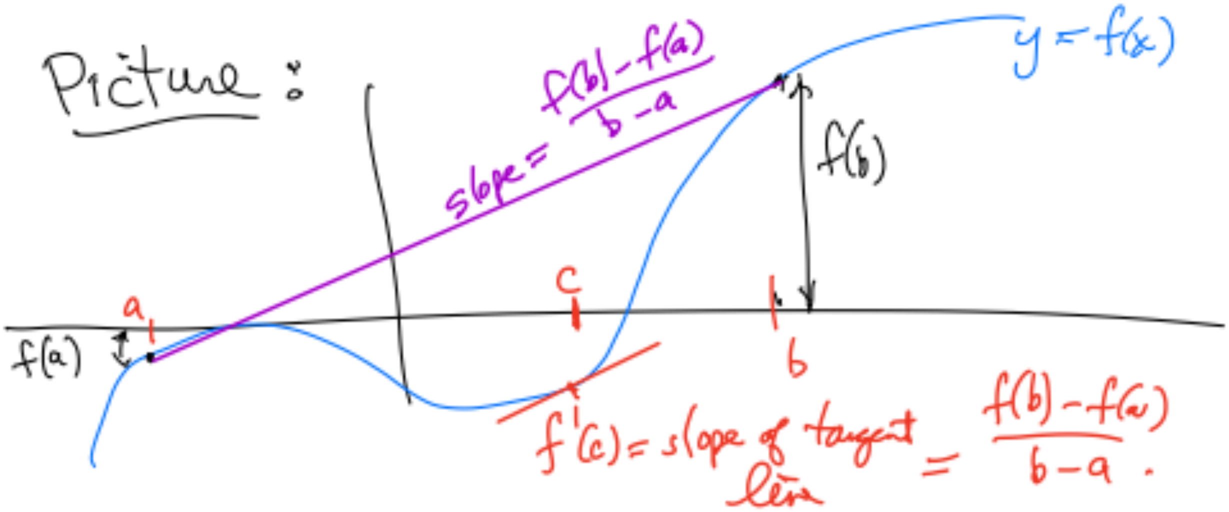
Proof: If $f(x)$ is constant on $[a,b]$, then $f'(x) = 0$ everywhere on (a,b), so any point $c \in (a,b)$ will work.

On the other hand, if f is not constant on $[a,b]$, then it must have an interior maximum or minimum inside (a,b), say at c. Then $f'(c) = 0$. ▨

Mean Value Theorem. If $f : [a,b] \to \mathbb{R}$ is a continuous function that is differentiable on (a,b), Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Picture:



Idea of proof: ① Equation of line connecting $(a, f(a))$ & $(b, f(b))$ is

$$y - f(a) = m(x-a), \quad m = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow y - f(a) = \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$$

$$y = \left(\frac{f(b)-f(a)}{b-a}\right)x - \frac{\left(a\cdot f(b) - a\cdot f(a)\right)}{b-a} + \cancel{f(a)}$$

$\frac{b f(a) - a f(b)}{b-a}$

$$\Rightarrow \boxed{y = \left(\frac{f(b)-f(a)}{b-a}\right)x + \frac{-a f(b) + b f(a)}{b-a}}$$

② Let $g(x) = f(x) - (\text{line equation})$

$$\Rightarrow g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)x + \frac{a f(b) - b f(a)}{b-a}$$

Now: $g(a) = g(b) = 0$ & g is diff'ble on (a,b)

and continuous on $[a,b]$. Now we apply Rolle's theorem, and we get there is a $c \in (a,b)$ so that

$$g'(c) = 0$$

$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}. \quad \blacksquare$$

FTC Fundamental Theorem of Calculus

Let $F : [a,b] \to \mathbb{R}$ that is differentiable, and suppose F' is a continuous function on $[a,b]$. Then $\int_a^b F'(x)\,dx$ exists and

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

<u>Notation:</u> $F(x)\Big|_a^b = F(b) - F(a)$.

i.e. FTC says $\int_a^b F'(x)\,dx = F(x)\Big|_a^b$

<u>Our example:</u>

$$\int_{-1}^{2} (3x^2-1)\,dx = ?$$

If $F(x) = x^3 - x$, then

$$F'(x) = 3x^2 - 1$$

$$\therefore \int_{-1}^{2} (3x^2-1)\,dx = \int_{-1}^{2} (x^3-x)'\,dx$$

$$\overset{FTC}{=\!=} x^3 - x \Big|_{(-1)}^{(2)}$$

$$= \left(2^3 - 2\right) - \left((-1)^3 - (-1)\right)$$

$$= 8 - 2 - (\underbrace{(-1)+1}_{0}) = \boxed{6}.$$ ✓

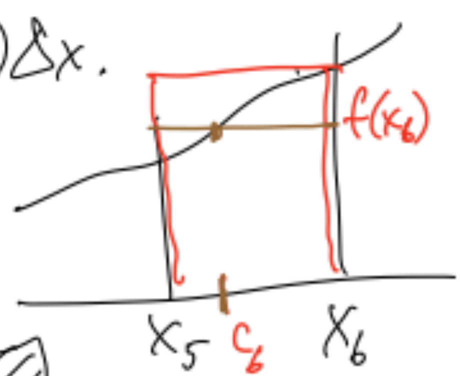
Proof of FTC:

$$F(b)-F(a) = F(\underset{x_n}{b}) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_3) - F(x_2) + F(x_2) - F(x_1) + F(x_1) - F(\underset{a}{x_0})$$

$$= \left(\frac{F(b)-F(x_{n-1})}{b-x_{n-1}}\right)\Delta x + \left(\frac{F(x_{n-1})-F(x_{n-2})}{x_{n-1}-x_{n-2}}\right)\Delta x + \cdots + \left(\frac{F(x_3)-F(x_2)}{x_3-x_2}\right)\Delta x + \left(\frac{F(x_2)-F(x_1)}{x_2-x_1}\right)\Delta x + \left(\frac{F(x_1)-F(x_0)}{x_1-x_0}\right)\Delta x$$

$$= F'(c_n)\Delta x + F'(c_{n-1})\Delta x + \cdots + F'(c_1)\Delta x$$

$$= \sum_{k=1}^{n} F'(c_k)\Delta x.$$



$$\Rightarrow F(b)-F(a) = \lim_{n\to\infty} \sum_{k=1}^{n} F'(c_k)\Delta x = \int_a^b F'(x)\,dx.$$